STABLE RAMSEY'S THEOREM AND MEASURE

DAMIR D. DZHAFAROV

Abstract. The stable Ramsey's theorem for pairs has been the subject of numerous investigations in mathematical logic. We introduce a weaker form of it by restricting from the class of all stable colorings to subclasses of it that are non-null in a certain effective measure-theoretic sense. We show that the sets that can compute infinite homogeneous sets for non-null many computable stable colorings and the sets that can compute infinite homogeneous sets for all computable stable colorings agree below $\emptyset'$ but not in general. We also answer the analogs of two well known questions about the stable Ramsey's theorem by showing that our weaker principle does not imply COH or WKL_0 in the context of reverse mathematics.

1. Introduction

The logical content of Ramsey's theorem has been studied extensively from the point of view of computability theory, beginning with the work of Jockusch [10]. Previous investigations, a partial survey of which can be found in [3], pp. 5–8, have been primarily concerned with identifying which complexity classes do or do not contain homogeneous sets for all computable colorings, thereby gauging the general difficulty of finding solutions to instances of Ramsey's theorem.

In this article, we concentrate on the *stable* form of Ramsey's theorem, which has played an important role in the study of Ramsey's theorem proper. We restrict our analysis from the class of all stable colorings to "large" or non-null subclasses of it, using a notion of nullity for Δ^0_2 sets (see Section 2). A previous result in this direction was obtained by Hirschfeldt and Terwijn [9, Theorem 3.1] and appears as Theorem 2.5 below. The focus here is on classifying properties of homogeneous sets of stable colorings not, as above, into those that are and are not universal, but into those that are and are not typical.

We begin by reviewing some of the terminology specific to the study of Ramsey's theorem. We refer the reader to Soare [20] for general background material on computability theory.

Definition 1.1. Let X be an infinite subset of ω and fix $n, k \in \omega$.

(1) $[X]^n$ denotes the set of all subsets of X of cardinality n.
(2) A *k-coloring of* $[X]^n$ is a map $f : [X]^n \to k$, where k is identified with the set of its predecessors, $\{0, \dots, k-1\}$.
(3) A set $H \subseteq X$ is *homogeneous* for f provided $f \upharpoonright [H]^n$ is constant.

2010 *Mathematics Subject Classification.* Primary 03D80, 05D10, 03D32, 03F35.

The author is grateful to his thesis advisers, Robert Soare, Denis Hirschfeldt, and Antonio Montalbán, for valuable insights and helpful comments during the preparation of this article. He also thanks the anonymous referee for a careful reading of the article, and for numerous comments that helped to improve it.

(4) If $X = \omega$ and $n = k = 2$, we call f simply a *coloring of pairs*, and if in addition $\lim_s f(x, s)$ exists for all x we call f a *stable coloring*.

Ramsey's theorem for pairs, denoted RT^2_2, asserts that every coloring of pairs has an infinite homogeneous set, while the stable Ramsey's theorem, denoted SRT^2_2, makes this assertion only for stable colorings. Restricting to computable colorings allows for the study of the effective content of homogeneous sets. For stable colorings, this reduces via the limit lemma to the study of infinite subsets and cosubsets (i.e., subsets of complements) of Δ^0_2 sets (for details, see [3], Lemma 3.5). In particular, every computable stable coloring has an infinite homogeneous set of degree at most $\mathbf{0}'$, a fact not true of computable colorings in general ([10], Corollary 3.2).

A natural question then is whether this upper bound can be improved somehow. With respect to the low$_n$ hierarchy, the following well-known results give a sharp separation.

Theorem 1.2 (Cholak, Jockusch, and Slaman [3], Theorem 3.1)**.** *Every computable coloring of pairs (not necessarily stable) has a low$_2$ infinite homogeneous set.*

Theorem 1.3 (Downey, Hirschfeldt, Lempp, and Solomon [6])**.** *There exists a computable stable coloring with no low infinite homogeneous set.*

The next result gives instead an improvement over the original bound with respect to the arithmetical hierarchy.

Theorem 1.4 (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [8], Corollary 4.6)**.** *Every computable stable coloring has an infinite homogeneous set of degree strictly below $\mathbf{0}'$.*

The above mentioned result of Hirschfeldt and Terwijn from [9] is a measure-theoretic analysis of Theorem 1.3 and shows that this theorem is atypical in that the collection of computable stable colorings that actually do have a low infinite homogeneous set is not null in the sense of Δ^0_2 nullity.

In this article, we similarly analyze Theorems 1.2 and 1.4. As both theorems are positive, we turn our attention to uniformity. Mileti [16, Theorem 5.3.7 and Corollary 5.4.6] showed that neither of these theorems admits a uniform proof. In Section 3, we extend one of his results by showing the following:

Theorem 1.5. *For each $\mathbf{d} < \mathbf{0}'$, the class of computable stable colorings having an infinite homogeneous set of degree at most $\mathbf{d}$ is Δ^0_2 null.*

In Section 4, we prove the following theorem showing that uniformity results can differ between the class of all computable stable colorings and more general subclasses of it that are not Δ^0_2 null. The Δ^0_3 bound also gives a partial result in the direction of showing that $< \mathbf{0}'$ in the preceding theorem cannot be replaced by low$_2$.

Theorem 1.6. *There is a degree $\mathbf{d} \leq \mathbf{0}''$ such that the class of computable stable colorings having an infinite homogeneous set of degree at most $\mathbf{d}$ is not Δ^0_2 null but is not equal to the class of all such colorings.*

In Section 5, we introduce several combinatorial principles related to SRT^2_2 from a measure-theoretic viewpoint, and study these in the context of reverse mathematics. In particular, we introduce the principle ASRT^2_2 which asserts that "non-negligibly many", rather than all, computable stable colorings admit a homogeneous set, and show that it lies strictly in between SRT^2_2 and the axiom DNR, and that it does not imply WKL_0. For background on reverse mathematics, see Simpson [19].

2. Δ^0_2 MEASURE

Martin-Löf introduced the definition of 1-randomness as a constructive notion of nullity. A stricter approach is that of Schnorr [17], which we now briefly recall.

Definition 2.1. A *martingale* is a function $M : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ that satisfies, for every $\sigma \in 2^{<\omega}$, the averaging condition

$$2M(\sigma) = M(\sigma 0) + M(\sigma 1). \tag{2.1}$$

We say that M *succeeds* on a set A if $\limsup_{n\to\infty} M(A \upharpoonright n) = \infty$, and we let the *success set* of M, $S[M]$, be the class of all sets on which M succeeds.

Unless otherwise noted, we shall assume that all our martingales are rational-valued, so that it makes sense to speak of martingales being computable. A class $\mathscr{C} \subseteq 2^\omega$ is said to be *computably null* if there is a computable martingale M which succeeds on each $A \in \mathscr{C}$, and *Schnorr null* if in fact there is a computable nondecreasing unbounded function h with $\limsup_{n\to\infty} \frac{M(A\upharpoonright n)}{h(n)} = \infty$ for every such A (i.e., the martingale succeeds sufficiently fast). The motivation here comes from the following classical result of Ville. The interested reader may wish to consult [22], Section 1.5, for a thorough treatment of effective measure, and [5] for background on algorithmic complexity.

Theorem 2.2 (Ville's theorem)**.** *A class $\mathscr{C} \subseteq 2^\omega$ has Lebesgue measure 0 if and only if there is martingale M such that $\mathscr{C} \subseteq S[M]$.*

By relativizing computable nullity to $\emptyset'$, we thus obtain a notion of nullity for the class of Δ^0_2 sets.

Definition 2.3. A class $\mathscr{C} \subseteq 2^\omega$ is Δ^0_2 *null* (or has Δ^0_2 *measure* 0) if there exists a Δ^0_2 martingale M such that $\mathscr{C} \subseteq S[M]$.

The study of this notion of nullity has been conducted principally by Terwijn [22, 23] and by Terwijn and Hirschfeldt [9], though in more general contexts it goes back to Schnorr (see [17], p. 55). It is a reasonable notion of nullity in that many of the basic properties one would expect to hold, do.

Proposition 2.4 (Lutz, see [22], Section 1.5)**.**

1. *The class of all Δ^0_2 sets is not Δ^0_2 null.*
2. *For every Δ^0_2 set A, $\{A\}$ is Δ^0_2 null.*
3. *If $\mathscr{C}_0, \mathscr{C}_1, \ldots$ is a sequence of subsets of 2^ω and $M_0, M_1, \ldots$ a uniformly Δ^0_2 sequence of martingales such that $\mathscr{C}_e \subseteq S[M_e]$ for every $e \in \omega$, then $\bigcup_{e\in\omega} \mathscr{C}_e$ is Δ^0_2 null.*

Additionally, Lutz and Terwijn (see [22], Theorem 6.2.1) have shown that for every Δ^0_2 set $A >_T \emptyset$, the upper cone $\{B : B \geq_T A\}$ is Δ^0_2 null, thereby effectivizing the corresponding classical result of Sacks for Lebesgue measure.

In view of the remarks following Definition 1.1, we can use Δ^0_2 nullity as a reasonable notion of "smallness" for computable stable colorings. It is easy to show that the class of Δ^0_2 sets having an infinite computable subset or cosubset is Δ^0_2 null, meaning that "most" stable colorings do not have a computable infinite homogeneous set (it is equally easy to extend this from computable to c.e. or even co-c.e.). The following result is an instance where the measure-theoretic approach differs from the classical computability-theoretic one.

Theorem 2.5 (Hirschfeldt and Terwijn [9], Theorem 3.1)**.** *The class of low sets is not Δ^0_2 null.*

In fact, the proof of the above theorem gives the stronger result that the class of Δ^0_2 sets not having an infinite low subset or cosubset is Δ^0_2 null. It follows that "most" computable stable colorings do not satisfy Theorem 1.3.

We will need a more uniform version of the above theorem, which we present in the form of the proposition below, in our proof of Theorem 1.6 in Section 4. It will rely on the following three facts. The first is the existence of a universal oracle c.e. martingale, i.e., of a real-valued martingale U such that for all sets X, $\{x \in \mathbb{Q} : x < U^X(\sigma)\}$ is X-c.e. uniformly in σ, and $S[U^X] = \{B \in 2^\omega : B \text{ not } X\text{-random}\}$ (see, e.g., [5], Corollary 5.3.5). By the proof of Proposition 1.5.5 in [22], we can fix a $u \in \omega$ so that for all X, $\Phi^{X'}_u$ is a rational-valued martingale with $S[\Phi^{X'}_u] \supseteq S[U^X]$. The second, which we will use repeatedly in the sequel, is van Lambalgen's theorem (see [5], Theorem 5.9.1), which states that a set is 1-random if and only if its odd and even halves are relatively 1-random. And the third fact, due to Nies and Stephan (unpublished, see [4], Theorem 3.4), is the following theorem. Recall that if $\{C_s\}_{s\in\omega}$ is a computable approximation of a Δ^0_2 set, its *modulus of convergence* of is the function $m(x) = (\mu s)(\forall t \geq s)[C_s(x) = C_t(x)]$. We write φ^X_e for the use of a computation Φ^X_e.

Theorem 2.6 (Nies and Stephan)**.** *Let C and B be sets such that C is Δ^0_2 and B-random (i.e., 1-random relative to B). If m is the modulus of convergence of a computable approximation of C, then $\varphi^B_x(x) \leq m(x)$ for all large enough x such that $\Phi^B_x(x)\downarrow$. In particular, since $m \leq_T \emptyset'$, B is GL_1 (i.e., $B' \leq_T B \oplus \emptyset'$).*

Recall that a Δ^0_2 *index* for a Δ^0_2 set A (or, more generally, for a partial $\emptyset'$-computable function f) is an $i \in \omega$ such that $A = \Phi^{\emptyset'}_i$ ($f = \Phi^{\emptyset'}_i$). A *lowness index* for a low set L is a Δ^0_2 index for L'. We draw attention to our use of $\Phi^X_{e,s}(x)$ to indicate a computation with oracle X run for s steps on input x, versus our use of $\Phi^X_e(x)[s]$ to indicatee the computation $\Phi^{X_s}_{e,s}(x)$ under the assumption of a fixed computable approximation (or enumeration) $\{X_s\}_{s\in\omega}$ of X. In particular, determining whether $\Phi^X_{e,s}(x)$ converges is X-computable, while for $\Phi^{X_s}_{e,s}(x)$ it is computable. We fix a computable enumeration $\{\emptyset'_s\}_{s\in\omega}$ of $\emptyset'$.

Proposition 2.7. *There exists a $\emptyset''$-computable function f such that for every $e, i \in \omega$, if $\Phi^{\emptyset'}_e$ is total and a martingale, and if i is a lowness index for some set L, then there is a set $B \notin S[\Phi^{\emptyset'}_e]$ such that $f(e,i)$ is a lowness index for $L \oplus B$.*

Proof. Fix $e, i \in \omega$ and let $u \in \omega$ be as described above. We define a partial $\emptyset'$-computable function $M : 2^{<\omega} \to \mathbb{Q}^{\geq 0}$. Given $\sigma \in 2^{<\omega}$, let $\widetilde{\sigma}$ be either λ if $\sigma = \lambda$, or $\sigma(0)\sigma(2)\cdots\sigma(2m)$ if σ has length $2m+1$ or $2m+2$ for some $m \geq 0$. If there exist $q, r \in \mathbb{Q}^{\geq 0}$ and $\tau \in 2^{<\omega}$ such that

(1) $\Phi^{\emptyset'}_e(\widetilde{\sigma})\downarrow= q$,
(2) $\Phi^{\emptyset'}_i(x)\downarrow= \tau(x)$ for all $x < |\tau|$ and $\Phi^{\tau}_u(\sigma)\downarrow= r$,

then let $M(\sigma) = \frac{1}{2}(q + r)$, and otherwise let $M(\sigma)$ be undefined. It is not difficult to see that M satisfies the averaging condition (2.1) where defined.

We next define $\{0,1\}$-valued partial $\emptyset'$-computable functions A, B, and C as follows. Given x, let

$$A(x) = \begin{cases} 0 & \text{if } M((A \upharpoonright x)\,0) \downarrow \leq M(A \upharpoonright x) \downarrow \\ 1 & \text{if } M((A \upharpoonright x)\,0) \downarrow > M(A \upharpoonright x) \downarrow \\ \uparrow & \text{otherwise} \end{cases}.$$

Then let $B(x) = A(2x)$ and $C(x) = A(2x+1)$ for all x, and let c be a Δ^0_2 index for C. Finally, define also $m_C(x) = (\mu s)(\forall t \geq s)[\Phi^{\emptyset'}_c(x)[t] \downarrow = \Phi^{\emptyset'}_c(x)[s] \downarrow]$.

Notice that if $\Phi^{\emptyset'}_e$ is a total martingale and $\Phi^{\emptyset'}_i$ is (the characteristic function of) the jump of some set L, then M is a Δ^0_2 martingale whose success set includes that of $\Phi^{L'}_u$, and A is a Δ^0_2 set on which M does not succeed. We then also have that $A = B \oplus C$, and it is readily seen from the definition of M that $B \notin S[\Phi^{\emptyset'}_e]$. Now because $A \notin S[M]$, A must be L-random, and so by van Lambalgen's theorem relative to L, C must be $L \oplus B$-random. Moreover, m_C is in this case the modulus of convergence for the computable approximation $\{C_s\}_{s\in\omega}$ of C defined by $C_s(x) = i$ if $\Phi^{\emptyset'}_c(x)[s] \downarrow = i$ and $C_s(x) = 0$ otherwise. Hence, by Theorem 2.6 (with $L \oplus B$ in place of B), there must be an n so that for all $x \geq n$, whenever $\varphi^{L\oplus B}_x(x)$ is defined it is bounded by $m_C(x)$.

Now to define $f(e,i)$, choose $j \in \omega$ so that $\Phi^{X'}_j = X$ for all sets X, and let h be a computable function so that for all $x \in \omega$, $x \in X$ if and only if $h(x) \in X'$. Using a $\emptyset''$ oracle, we search for the first of the following to occur:

(1) $\Phi^{\emptyset'}_e$ is undefined or does not satisfy the averaging condition (2.1) on some string,
(2) $\Phi^{\emptyset'}_i$ is undefined on some number,
(3) there exist a $\sigma \in 2^{<\omega}$ and an $x < |\sigma|$ such that $\Phi^{\emptyset'}_i(h(y)) \downarrow = \sigma(y)$ for all $y < |\sigma|$, and either $\Phi^{\sigma}_x(x) \downarrow$ and $\Phi^{\emptyset'}_i(x) \downarrow = 0$, or else $\Phi^{\tau}_x(x) \uparrow$ for all $\tau \supseteq \sigma$ and $\Phi^{\emptyset'}_i(x) \downarrow = 1$,
(4) there is an $n \in \omega$ so that for all σ, τ of the same length and all $x \geq n$, if
    (a) $\Phi^{\emptyset'}_i(h(y)) \downarrow = \sigma(y)$ for all $y < |\sigma|$,
    (b) $B(y) \downarrow = \tau(y)$ for all $y < |\tau|$,
    (c) $\Phi^{\sigma\oplus\tau}_x(x) \downarrow$ and $m_C(x) \downarrow$,
    then $\varphi^{\sigma\oplus\tau}_x(x) \leq m_C(x)$.

This search necessarily terminates, for if (1), (2), and (3) above do not obtain, then we are precisely in the situation of the preceding paragraph, so (4) must obtain as discussed there. If (1), (2), or (3) occur, let $f(e,i) = 0$. Otherwise, choose the least n witnessing the occurrence of (4) and let $f(e,i)$ be a Δ^0_2 index, found according to some fixed effective procedure, for the following function. On input x, the function waits for $m_C(x)$ to converge, then chooses the smallest $y \geq n$ such that $\Phi^X_x = \Phi^X_y$ for all sets X and searches for the first σ, τ of the same positive length so that (a) and (b) in (4) above hold. It then outputs 1 or 0 depending as $\Phi^{\sigma\oplus\tau}_y(x) \downarrow$ with use bounded by $m_C(x)$ or not. $\square$

3. Almost s-Ramsey degrees

In [3, Sections 4 and 5], Cholak, Jockusch, and Slaman give two proofs of Theorem 1.2 for the stable case, but neither of them is uniform over the stable colorings (see the discussion at the beginning of Section 12.3 of [3]), and similarly in the

case of the proof of Theorem 1.4. To address whether such nonuniformities were essential, Mileti introduced the following class of degrees:

Definition 3.1 (Mileti [16], Definition 5.1.2)**.** A Turing degree $\mathbf{d}$ is *s-Ramsey* if every Δ^0_2 set has an infinite subset or cosubset of degree at most $\mathbf{d}$.

Obviously, an s-Ramsey degree can also be defined as one which bounds the degree of a homogeneous set for every computable stable coloring. Thus, the following results imply that Theorems 1.2 and 1.4 do not have uniform proofs.

Theorem 3.2 (Mileti [16], Theorem 5.3.7 and Corollary 5.4.6)**.**

(1) *The only Δ^0_2 s-Ramsey degree is $\mathbf{0}'$.*
(2) *There is no low$_2$ s-Ramsey degree.*

With the definition of Δ^0_2 nullity in hand, we can generalize s-Ramsey degrees by passing from the class of all Δ^0_2 sets to subclasses of it which are not Δ^0_2 null.

Definition 3.3. A Turing degree $\mathbf{d}$ is *almost s-Ramsey* if the collection of Δ^0_2 sets with an infinite subset or cosubset of degree at most $\mathbf{d}$ is not Δ^0_2 null.

We obtain the same class of degrees in the above definition whether we insist on considering cosubsets or not. For if a martingale M succeeds on the class of all Δ^0_2 sets having an infinite subset of degree at most $\mathbf{d}$, then the martingale $M + N$, where $N(\sigma) = M((1 - \sigma(0))(1 - \sigma(1)) \cdots (1 - \sigma(|\sigma| - 1)))$ for all σ, succeeds on the class of all Δ^0_2 sets having an infinite such subset or cosubset. This is in stark contrast to Definition 3.1 even if we deal only with infinite, coinfinite Δ^0_2 sets, as it is easy to construct such a set so that all of its infinite subsets compute $\emptyset'$ (in fact, for any infinite set A, if B is the set of all prefixes of A under some fixed computable bijection of $2^{<\omega}$ with ω, then each infinite subset of B computes A).

The preceding definition was suggested by D. Hirschfeldt, who asked whether Mileti's results still hold if s-Ramsey degrees are replaced by the weaker almost s-Ramsey degrees, and more generally, whether the two classes of degrees are the same. Theorem 1.5, stated in Section 1 and restated in terms of almost s-Ramsey degrees below, is an affirmative answer with regards to the analog of Theorem 3.2 (1). We discuss the other questions, and give a separation of s-Ramsey and almost s-Ramsey degrees, in the next section.

Theorem 1.5. *The only Δ^0_2 almost s-Ramsey degree is $\mathbf{0}'$.*

Proof. Fix a set $D <_T \emptyset'$. For each $e \in \omega$, we construct uniformly in $\emptyset'$ a martingale M_e so as to satisfy the requirement

$$R_e : (\exists^\infty x)(\forall y \leq x)[\Phi^D_e(y) \downarrow \in \{0, 1\} \wedge \Phi^D_e(x) = 1] \rightarrow (\forall A \supseteq \Phi^D_e)[A \in S[M_e]].$$

By Theorem 2.4 (3)—letting $\mathscr{C}_e$ there be $\{A : A \supseteq \Phi^D_e\}$ if Φ^D_e is a characteristic function and $\emptyset$ otherwise—this will ensure that the collection of sets containing an infinite subset computable in D is Δ^0_2 null, and hence, by the remarks following Definition 3.3, that $\deg(D)$ is not almost s-Ramsey.

Fix a total increasing function $f \leq_T \emptyset'$ not dominated by any function of degree strictly below $\mathbf{0}'$. We define M_e by stages, at stage s defining M_e on all strings of length t for a specific $t \geq s$.

Stage $s = 0$. Let $M_e(\lambda) = 1$.

Stage $s + 1$. Assume M_e has been defined on all strings of length t for some $t \geq s$. Search $\emptyset'$-computably for a string $\tau \subseteq D$ and a number $x \geq t$ such that $|\tau|, x \leq f(t)$

and $\Phi^{\tau}_{e,|\tau|}(x)\downarrow= 1$. If the search succeeds, choose the least x for which it does so. Then for each $\sigma \in 2^{<\omega}$ of length t, and for all $\tau \supset \sigma$ with $|\tau| \leq x+1$, define

$$M_e(\tau) = \begin{cases} M_e(\sigma) & \text{if } |\tau| \leq x \\ 2M_e(\sigma) & \text{if } |\tau| = x+1 \wedge \tau(x) = 1 \\ 0 & \text{if } |\tau| = x+1 \wedge \tau(x) = 0 \end{cases}.$$

Otherwise, set $M_e(\sigma 0) = M_e(\sigma 1) = M_e(\sigma)$ for all σ of length t.

It is clear that the construction succeeds in defining M_e on all of $2^{<\omega}$. To verify that R_e is met, suppose that Φ^D_e is the characteristic function of an infinite set. Then the function

$$g(y) = (\mu s)(\exists x \geq y)(\forall z < x)[\Phi^D_{e,s}(x)\downarrow= 1 \wedge (y \leq z \rightarrow \Phi^D_{e,s}(z)\downarrow= 0)]$$

is total and computable in D, so by choice of f there must exist infinitely many y such that $g(y) \leq f(y)$. Fix $A \supseteq \Phi^D_e$ and suppose that at the end of some stage s' of the construction, $M_e(A \upharpoonright t)$ for some $t \geq 0$ is defined and positive, while $M_e(A \upharpoonright t+1)$ is not yet defined. Choose the least $y \geq t$ such that $g(y) \leq f(y)$. If f is replaced by g in the search performed at each stage of the construction, then the search always succeeds, so it must necessarily succeed at some stage $s > s'$. Fix the least such s. Then by construction, at every stage between s' and s, M_e gets defined only on the successors of the longest strings it was defined on at the previous stage, and it is given the same value on these successors. In particular, at the beginning of stage s, we have that M_e is defined on $A \upharpoonright t + (s-s') - 1$ at the start of stage s, and $M_e(A \upharpoonright t + (s-s') - 1) = M_e(A \upharpoonright t)$. By choice of s, there exists a string $\tau \subseteq D$ and a number $x \geq t + (s-s') - 1$ such that $|\tau|, x \leq f(t)$ and $\Phi^{\tau}_{e,|\tau|}(x)\downarrow= 1$. Then at stage s, M_e gets defined on $A \upharpoonright x+1$ with $M_e(A \upharpoonright y) = M_e(A \upharpoonright t)$ for all $y \leq x$ and, since $A(x) = \Phi^D_e(x) = 1$, $M_e(A \upharpoonright x+1) = 2M_e(A \upharpoonright t)$. Since $x+1 > t$, it follows that $\limsup_n M_e(A \upharpoonright n) = \infty$. □

We illustrate an application of the preceding theorem by briefly looking at the Muchnik degrees of classes of infinite subsets and cosubsets of Δ^0_2 sets. Recall that if $\mathscr{A}$ and $\mathscr{B}$ are classes of sets, we say $\mathscr{A}$ is Muchnik (or weakly) reducible to $\mathscr{B}$, and write $\mathscr{A} \leq_w \mathscr{B}$, if every element of $\mathscr{B}$ computes an element of $\mathscr{A}$; if also $\mathscr{B} \leq_w \mathscr{A}$, we write $\mathscr{A} \equiv_w \mathscr{B}$. We refer the reader to Binns and Simpson [2], Section 1, for additional background.

Definition 3.4. Given a Δ^0_2 set A, let $H(A)$ be the collection of all infinite subsets or cosubsets of A, and for a class $\mathscr{C}$ of Δ^0_2 sets let $H(\mathscr{C})$ denote the structure $\{H(A) : A \in \mathscr{C}\}$ under $\leq_w$. Given a computable stable coloring f, let $H(f)$ be the collection of all infinite homogeneous sets of f.

Clearly, for each Δ^0_2 set A there is a computable stable f with $H(A) \equiv_w H(f)$, and conversely. Thus, we may use the two notions interchangeably here.

Proposition 3.5. *$H(\Delta^0_2)$ is a lower semilattice.*

Proof. Given two stable colorings, f_0 and f_1, we define a third, f, such that $H(f) \equiv_w H(f_0) \cup H(f_1)$. For $x, y \in \omega$, let $f(2x, y)$ equal $f_0(x, z)$ for the least z such that $2z \geq y$, and let $f(2x+1, y)$ equal $f_1(x, z)$ for the least z such that $2z+1 \geq y$. It is easy to see that f is stable.

If H is an infinite homogeneous set for f_0, respectively for f_1, then the set $\{2x : x \in H\}$, respectively $\{2x+1 : x \in H\}$, is homogeneous for f, implying that

$H(f) \leq_w H(f_0) \cup H(f_1)$. Conversely, let H be any infinite homogeneous set for f and let $H_0 = \{x : 2x \in H\}$ and $H_1 = \{x : 2x+1 \in H\}$. One of H_0 and H_1, say H_i, must be infinite, and this set is clearly computable in H and homogeneous for f_i, implying that $H(f_0) \cup H(f_1) \leq_w H(f)$. □

Notice that if there were a largest element in $H(\Delta^0_2)$, it would have an infinite homogeneous set $H <_T \emptyset'$ by Theorem 1.4. Then $\deg(H)$ would be an s-Ramsey degree $< \mathbf{0}'$, contrary to part (1) of Theorem 3.2. This yields the following:

Corollary 3.6 (Mileti [16], Corollary 5.4.8)**.** *There is no largest element in $H(\Delta^0_2)$.*

Using Theorem 1.5, we can now extend this result as follows.

Corollary 3.7. *If $\mathscr{C}$ is a class of Δ^0_2 sets that is not Δ^0_2 null, then there is no largest element in $H(\mathscr{C})$.*

For general interest, we remark that the algebraic properties of the structure $H(\Delta^0_2)$ have not previously been studied. It can be shown, though we do not elaborate on it here, that there are no maximal elements in it, and that for every finite collection of elements in it there is an element incomparable with each of them (proofs will appear in [7]). Beyond this, little is known; in particular, we do not know the answer to the following question:

Question 3.8. Is $H(\Delta^0_2)$ elementarily equivalent to $H(\mathscr{C})$ for every class $\mathscr{C}$ of Δ^0_2 sets that is not Δ^0_2 null?

4. An almost s-Ramsey degree that is not s-Ramsey

In this section, we give a proof of Theorem 1.6, restated equivalently below, thereby showing that the s-Ramsey degrees are a proper subclass of the almost s-Ramsey degrees. We do not know whether the analog of Theorem 3.2 (2) holds for almost s-Ramsey degrees, but as every low$_2$ degree is Δ^0_3, our result is a partial step towards a negative answer.

Theorem 1.6. *There is a Δ^0_3 almost s-Ramsey degree that is not s-Ramsey.*

Proof. Fix a Δ^0_2 set A with no low infinite subset or cosubset. Computably in $\emptyset''$, we construct a set D and infinite low sets $L_0, L_1, \ldots$ that satisfy, for every $e \in \omega$ and $i < 2$, the requirements

$$R_e \quad : \quad L_e \times \{e\} =^* D^{[e]} \wedge (\Phi^{\emptyset'}_e \text{ is a total martingale} \rightarrow L_e \notin S[\Phi^{\emptyset'}_e]),$$

$$S_{e,i} \quad : \quad \Phi^D_e \text{ is total, } \{0,1\}\text{-valued and infinite} \rightarrow (\exists x)[\Phi^D_e(x) = 1 \wedge A(x) = i].$$

The first set of requirements ensures that $\{L_e : e \in \omega\}$ is not Δ^0_2 null and that $L_e \leq_T D$ for all e, and the second that no infinite subset or cosubset of A is computable in D. Hence, $\deg(D)$ will be the desired degree.

We let $D = \bigcup_s D_s$, where $D_0, D_1, \ldots$ are constructed in stages as follows. At stage s, we define a finite set D_s, a function f_s with domain ω, and for each e a restraint $r_{e,s}$. We also declare each requirement either *online* or *offline*. Let h be a computable function such that for all sets X and all $x \in \omega$, $x \in X$ if and only if $h(x) \in X'$.

Construction.

Stage $s = 0$. Set $D_0 = \emptyset$, and $f_0(e) = r_{e,s} = 0$ for all e. Declare all requirements R_e and $S_{e,i}$ for $e \in \omega$ and $i < 2$ online.

Stage $s+1$. Let D_s, f_s, and $r_{0,s}, r_{1,s}, \dots$ be given. Assume inductively that cofinitely many requirements are still online, and that the value of f_s is 0 on cofinitely many arguments.

Case 1: $s+1 \equiv 0 \bmod 3$ *or* $s+1 \equiv 1 \bmod 3$. Suppose $s+1 = 3\langle e,j\rangle + i$, where $e, j \in \omega$ and $i < 2$. If $S_{e,i}$ is online, ask whether there exists an $x \in \omega$ and a finite set F such that

(1) $D_s \subseteq F \subseteq D_s \cup \{\langle y, e'\rangle \geq r_{e,s} : e' \leq e \to R_{e'} \text{ online}\}$,
(2) $\Phi_e^F(x) \downarrow= 1$ and $A(x) = i$,
(3) for $e' \leq e$ with $R_{e'}$ online and all $\langle y, e'\rangle \leq \max F \cup \{z : z \leq \varphi_e^F(x)\}$, $\Phi^{\emptyset'}_{f_s(e')}(h(2y+1)) \downarrow$, and if $\langle y, e'\rangle \in F - D_s$ then $\Phi^{\emptyset'}_{f_s(e')}(h(2y+1)) = 1$.
(4) for $e' \leq e$ with $R_{e'}$ online and all $\langle y, e'\rangle \leq \varphi_e^F(x)$, if $\langle y, e'\rangle \notin F - D_s$ then $\Phi^{\emptyset'}_{f_s(e')}(h(2y+1)) = 0$.

If so, we find the first such F and x in some fixed enumeration, set $D_{s+1} = F$, let $r_{e',s+1} = r_{e',s}$ for $e' < e$, and let $r_{e',s+1}$ be the least number greater than $\max\{r_{e'',s} : e \leq e'' \leq e'\}$ and $\varphi_e^F(x)$ for $e' \geq e$. We say that $S_{e,i}$ *acts* at stage $s+1$, declare it offline, and declare all $S_{e',i}$ with $e' > e$ currently offline online again. Otherwise, or if $S_{e,i}$ is already offline, we set $D_{s+1} = D_s$ and $r_{e',s+1} = r_{e',s}$ for all e'. Either way, we let $f_{s+1} = f_s$. Notice that the question of whether or not x and F in Case 1 exist is $\Sigma_1^{0,\emptyset'}$, and hence can be answered by $\emptyset''$.

Case 2: $s+1 \equiv 2 \bmod 3$. We begin by choosing the least e such that R_e is online and $f_s(e') = 0$ for all $e' \geq e$, which must exist by inductive hypothesis. Set $r_{e',s+1} = r_{e',s}$ for all e'. Fix $e' \in \omega$ and assume we have defined f_{s+1} on all $e'' < e'$. If $e' > e$ or if $R_{e'}$ is offline, set $f_{s+1}(e') = 0$. Otherwise, let i be either a fixed lowness index for $\emptyset$ if there is no $e'' < e'$ such that $R_{e''}$ is online, or else $f_{s+1}(e'')$ for the greatest such e''. Then let $f_{s+1}(e')$ be the result of applying to e' and i the $\emptyset''$-computable function asserted to exist by Proposition 2.7.

To define D_{s+1}, begin by letting $D^{[e']}_{s+1} = D^{[e']}_s$ for all e' such that at least one of the following holds:

(1) $e' > e$,
(2) $R_{e'}$ is offline,
(3) $\Phi^{\emptyset'}_{f_{s+1}(e')}$ is not defined or not $\{0,1\}$-valued on $h(2x+1)$ for some $x \leq s$,
(4) $\Phi^{\emptyset'}_{e'}$ is not defined or does not satisfy the averaging condition (2.1) on some string of length $\leq s$,

For all e' for which (4) obtains, declare $R_{e'}$ offline, and declare all offline $S_{e'',i}$ requirements for $e'' \geq e'$ online. For all e' such that none of the above obtain, let $D^{[e']}_{s+1} = D^{[e']}_s \cup \{\langle x, e'\rangle > r_{e',s+1} : x \leq s \wedge \Phi^{\emptyset'}_{f_{s+1}(e')}(h(2x+1)) \downarrow= 1\}$.

In either case above only finitely many requirements are declared offline, and f_{s+1} is defined to be positive on only finitely many elements. Thus, the induction can continue.

End construction.

The entire construction can be performed using a $\emptyset''$ oracle, hence $D \leq_T \emptyset''$. We now verify that all requirements are satisfied. To begin, note that each R requirement can only switch from being online to being offline but not back, and each $S_{e,i}$ requirement, once offline, can only become online again because some $R_{e'}$

requirement with $e' \leq e$ has become offline. In particular, each S requirement acts at most finitely many times. Since for every e, $r_{e,s}$ is a nondecreasing function in s that increases only when some $S_{e',i}$ with $e' \leq e$ acts, $\lim_s r_{e,s}$ exists.

Claim 4.1. *For every $e \in \omega$, $f(e) = \lim_s f_s(e)$ exists. Moreover, if R_e is permanently online then $f(e)$ is a lowness index, and if R_e is not permanently online then $f(e) = 0$ and $D^{[e]}$ is finite.*

Proof. Fix $e \in \omega$ and assume the claim holds for all $e' < e$. Fix a stage $s \geq 0$ such that for all $e' \leq e$ and all $i < 2$,

(1) if $e' < e$ then $f(e') \downarrow= f_t(e')$ for all $t > s$,
(2) if $R_{e'}$ is cofinitely often offline, then it is offline at all stages $t \geq s$,
(3) if $S_{e',i}$ is cofinitely often offline, then it is offline at all stages $t \geq s$.

First suppose R_e is online at stage s, and hence permanently thereafter. Since 0 is not a lowness index (we assume an enumeration of oracle machines, such as the standard one based on Gödel numberings, that makes this true), the inductive hypothesis implies that at any stage $t \geq s$ that is congruent to 2 modulo 3, the number chosen at the beginning of Case 2 of the construction is at least as big as e. Hence, we see from the construction that the value of $f_t(e)$ at any stage $t \geq s$ depends only on e and, if there is an $R_{e'}$ with $e' < e$ which is online at stage s, on $f_t(e') = f(e')$ for the largest such e'. Thus $f_t(e)$ has the same value for all $t \geq s$, so $f(e) = f_s(e)$.

As R_e is never declared offline, it must be that condition (4) in Case 2 of the construction never occurs, and hence that $\Phi_e^{\emptyset'}$ is a total martingale. Let L be either $\emptyset$ or, if there exists an $e' < e$ with $R_{e'}$ permanently online, $\Phi_{f(e')}^{\emptyset'}$ for the greatest such e'. Then it follows by construction and by Proposition 2.7 that $f(e)$ is a lowness index for $L \oplus B$, where B is a set not in $S[\Phi_e^{\emptyset'}]$. In particular, $f(e)$ is a lowness index, as desired.

Now suppose R_e is offline at stage s. Then $f_t(e)$ is defined to be 0 at all stages $t \geq s$, so $f(e) = 0$. Now no elements can be put into $D_t^{[e]}$ at any stage $t > s$ under Case 1 of the construction, because by condition (1) in that case this can only be done because of the action of some requirement $S_{e',i}$ with $e' \leq e$, and all such requirements have stopped acting by stage s. Moreover, no elements can be put into $D_t^{[e]}$ under Case 2, because condition (2) in that case allows this only when R_e is still online. Hence, $D_t^{[e]} = D_s^{[e]}$ for all $t \geq s$, and so $D^{[e]}$ is finite. □

Claim 4.2. *For every $e \in \omega$, requirement R_e is satisfied via a set L_e such that $\bigoplus_{e' \leq e} L_{e'}$ is low.*

Proof. First suppose that $\Phi_e^{\emptyset'}$ is a total martingale. Then condition (4) in Case 2 of the construction never occurs and R_e is online at all stages. Let L be as in the proof of the preceding claim, and let L_e be the set B from there, so that $f(e)$ is a lowness index for $L \oplus L_e$ and $L_e \notin S[\Phi_e^{\emptyset'}]$.

It then remains only to show that $L_e \times \{e\} =^* D^{[e]}$. Let s be a stage as in the proof of the preceding claim. Since no $S_{e',i}$ requirement with $e' \leq e$ can act at any stage $t \geq s$, it follows by condition (3) in Case 1 of the construction, as well as the fact that $L_e = \{x : \Phi_{f(e)}^{\emptyset'}(h(2x+1)) \downarrow= 1\}$, that any element put into $D_t^{[e]}$ for the sake of an S requirement must belong to $L_e \times \{e\}$. For the same reason we must have that $r_e = r_{e,t}$ for any stage $t \geq s$, and, as mentioned in the previous claim, the

number chosen at the beginning of Case 2 of the construction at any such stage t cannot be smaller than e. Hence, at the end of every stage $t \geq s$ that is congruent to 2 modulo 3, all elements x in $L_e \times \{e\}$ with $r_e < x \leq t$ are put into $D_t^{[e]}$. It follows that $\{x \in D^{[e]} : x > \max D_s^{[e]}\} \subseteq L_e \times \{e\}$ and $\{x \in L_e \times \{e\} : x > r_e\} \subseteq D^{[e]}$, which yields the desired result.

Next suppose that $\Phi_e^{\emptyset'}$ is not a total martingale. Then at some stage, condition (4) in Case 2 of the construction occurs and R_e is declared offline. By the previous claim, $D^{[e]}$ is finite, so if we let $L_e = \emptyset$ then L_e is low and requirement R_e is met.

Finally, given e let $e_0 < e_1 < \cdots < e_n$ be a listing of all $e' \leq e$ such that $R_{e'}$ is online at stage s. Then $\bigoplus_{j \leq n} L_{e_j}$ is low, for $f(e_0)$ is a lowness index for $\emptyset \oplus L_{e_0}$, $f(e_1)$ is a lowness index for $(\emptyset \oplus L_{e_0}) \oplus L_{e_1}$, and so on. Hence $\bigoplus_{e' \leq e} L_{e'}$ is low since $L_{e'} = \emptyset$ for all $e' \neq e_j$ for any j, and this completes the proof. $\square$

Claim 4.3. *For every $e \in \omega$ and $i < 2$, $S_{e,i}$ is satisfied.*

Proof. Fix e and i and assume inductively that the claim holds for all $e' < e$. Fix a stage $s \geq 0$ congruent to i modulo 3 such that for all $e' \leq e$, $f_s(e') = f(e)$ and $D_s^{[e']} = D^{[e']}$ if $R_{e'}$ is not permanently online, and for all $e' < e$, $r_{e',s} = r_e$ and no $S_{e',i}$ requirement with $e' < e$ acts at or after stage s. Assume further that Φ_e^D is total, $\{0,1\}$-valued, and infinitely often takes the value 1, as otherwise $S_{e,i}$ is satisfied trivially. Since $L_{e'} \times \{e'\} =^* D^{[e']}$ for all $e' \leq e$, it follows by the previous claim that $\bigcup_{e' \leq e} D^{[e']}$ is low, and since D_s is finite, also that $\bigcup_{e' \leq e} D^{[e']} \cup D_s$ is low.

Now there must exist an $x \in \omega$ and a finite set F such that $A(x) = i$ and such that the following conditions hold:

1. $D_s \subseteq F \subseteq D_s \cup \{\langle y, e' \rangle \geq r_{e,s} : e' \leq e \to R_{e'} \text{ online}\}$,
2. $\Phi_e^F(x) \downarrow = 1$,
3. for all $e' \leq e$, $F^{[e']} \subseteq D^{[e']}$,
4. for all $e' \leq e$, $F^{[e']} \upharpoonright \varphi_e^F(x)) = D^{[e']} \upharpoonright \varphi_e^F(x)$.

Indeed, from our assumptions about Φ_e^D it follows that there exist arbitrarily large numbers x and corresponding finite sets F satisfying (1)–(4) above, for example all sufficiently long initial segments of D. And we can clearly find such x and F computably in $\bigcup_{e' \leq e} D^{[e']} \cup D_s$. Hence, if $A(x)$ were equal to $1 - i$ for all such x, then depending as i is 0 or 1, $\bigcup_{e' \leq e} D^{[e']} \cup D_s$ could compute an infinite subset or infinite cosubset of A, contradicting that A has no low infinite subset or cosubset.

By choice of s, it is easily seen that for all $e' \leq e$, all elements in $D^{[e']} - D_s$ belong to $L_{e'} \times \{e'\}$. It follows that the question about an $x \in \omega$ and a finite set F asked at stage s of the construction is precisely the question of whether there exist x and F satisfying the conditions above, and as such must have an affirmative answer. Hence $S_{e,i}$ acts, meaning that for some such x and F, $D_{s+1} = F$ and $r_{e',t}$ is greater than $\varphi_e^F(x)$ for all $t > s$ and all $e' \geq e$. No requirements can then ever put into D_t any elements below $\varphi_e^F(x)$ at any stage $t > s$, meaning that the $\Phi_e^F(x)$ computation is preserved and so $\Phi_e^D(x) = 1$. Consequently, requirement $S_{e,i}$ is satisfied. $\square$

$\square$

Question 4.4. Does there exist a low_2 almost s-Ramsey degree?

5. Almost stable Ramsey's theorem

The proof-theoretic strength of SRT^2_2, as a principle of second order arithmetic, was first studied by Cholak, Jockusch, and Slaman ([3], Sections 7 and 10). One major open problem is whether SRT^2_2 implies WKL_0 over RCA_0 (see [3], p. 53), the closest related result being by Hirschfeldt, et al. [8, Theorem 2.4] that SRT^2_2 implies the weaker axiom DNR. (That WKL_0 does not imply SRT^2_2 is by [3], Theorems 11.1 and 11.4; it can also be seen by Theorem 1.3 and the fact that WKL_0 has a model consisting entirely of low sets). Another question is whether SRT^2_2 implies COH, which is equivalent by Theorem 1.3 of [3] and the correction given in section A.1 of [16] to the question of whether SRT^2_2 implies RT^2_2. For completeness, we recall the definitions of DNR and COH.

Definition 5.1. The following definitions are made in RCA_0.

(1) COH is the statement that for every sequence $\langle X_i : i \in \mathbb{N}\rangle$ of sets, there is an infinite set X such that for every $i \in \mathbb{N}$, either $X \subseteq^* X_i$ or $X \subseteq^* \overline{X_i}$.

(2) DNR is the statement that for every set X there exists a function f that is DNR^X, i.e., such that for all $e \in \mathbb{N}$, $f(e) \neq \Phi^X_e(e)$.

In this section, we study several principles inspired by our investigations above and related to SRT^2_2 by means of a formal notion of Δ^0_2 nullity.

Definition 5.2. The following definitions are made in RCA_0.

(1) A *martingale approximation* is a function $M : 2^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}^{\geq 0}$ such that $\lim_s M(\sigma, s)$ exists for every $\sigma \in 2^{<\mathbb{N}}$ (i.e., $M(\sigma, s) = M(\sigma, t)$ for all sufficiently large $s, t \in \mathbb{N}$), and for all $s \in \mathbb{N}$,

$$2M(\sigma, s) = M(\sigma 0, s) + M(\sigma 1, s).$$

(2) We say M *succeeds on* a stable coloring $f : [\mathbb{N}]^2 \to 2$ if

$$(\forall n)(\exists \sigma)(\exists s)(\forall t \geq s)(\forall x < |\sigma|)[\sigma(x) = f(x, t) \wedge M(\sigma, t) = M(\sigma, s) > n]. \tag{5.1}$$

We can now state an "almost stable Ramsey's theorem", along with principles asserting the existence of s-Ramsey and almost s-Ramsey degrees.

Definition 5.3. The following definitions are made in RCA_0.

(1) ASRT^2_2 is the statement that for every martingale approximation M, there is a stable coloring $f \leq_T M$ on which M does not succeed and which has an infinite homogeneous set.

(2) SRAM is the statement that for every set X, there is a set Y as follows: every stable coloring $f \leq_T X$ has an infinite homogeneous set $H \leq_T Y$.

(3) ASRAM is the statement that for every set X, there is a set Y as follows: for every martingale approximation $M \leq_T X$ there is a stable coloring $f \leq_T X$ on which M does not succeed and which has an infinite homogeneous set $H \leq_T Y$.

Notice that the class of Δ^0_2 sets having an infinite subset or cosubset in a given ω-model of ASRT^2_2 is not Δ^0_2 null.

We begin with the following formalization of Proposition 2.4 (1). Recall that $\mathsf{B}\Pi^0_1$ is the collection of all statements of the form

$$\forall n[(\forall x < n)(\exists y)\varphi(x, y) \to (\exists m)(\forall x < n)(\exists y < m)\varphi(x, y)],$$

where φ is a Π^0_1 formula (we do not know if its use below can be avoided).

Lemma 5.4 ($\mathsf{RCA}_0 + \mathsf{B}\Pi^0_1$)**.** *For every martingale approximation M, there is a stable coloring $f \leq_T M$ on which M does not succeed.*

Proof. Let M be a martingale approximation, say with $\lim_s M(\lambda, s) = 1$. Then by Definition 5.2, if $M(\sigma, s) \leq 1$ for some s, either $M(\sigma 0, s) \leq 1$ or $M(\sigma 1, s) \leq 1$. Choose s_0 so that $M(\lambda, s) \leq 1$ for all $s \geq s_0$. For every x and $s \geq s_0$, a simple Σ^0_0 induction then shows that there exists $\sigma \in 2^{<\mathbb{N}}$ of length $x+1$ such that

$$(\forall y \leq x+1)[M(\sigma \upharpoonright y, s) \leq 1] \wedge (\forall y \leq x)[\sigma(y) = 1 \rightarrow M((\sigma \upharpoonright y)\, 0, s) > 1],$$

and that this string is unique. Define $f : [\mathbb{N}]^2 \rightarrow 2$ by letting $f(x,s)$ for $x < s$ be 0 or $\sigma(x)$ for the above σ depending as $s < s_0$ or $s \geq s_0$. Clearly, f has a Σ^0_0 definition with M as parameter, so $f \leq_T M$. We claim that f is stable and that M does not succeed on it. Fix x in $\mathbb{N}$ and using $\mathsf{B}\Pi^0_1$ choose an $s \geq s_0$ with $M(\sigma, t) = M(\sigma, s)$ for all $t \geq s$ and $\sigma \in 2^{<\mathbb{N}}$ of length $\leq x+1$. Then the σ used to define $f(x,s)$ will be same as that used to define $f(x,t)$ for all $t \geq s$. Hence, $f(x,t) = \sigma(x)$ for all $t \geq s$, and as $M(\sigma, t) \leq 1$ we have the negation of (5.1) holding with $n = 1$. □

Basic relations of implication and nonimplication between SRT^2_2 and the principles given in Definition 5.3 are established in the next proposition.

Proposition 5.5. *Over* RCA_0*,*

(1) $\mathsf{ACA}_0 \rightarrow \mathsf{SRAM} \rightarrow \mathsf{SRT}^2_2 \rightarrow \mathsf{ASRT}^2_2$ *and* $\mathsf{SRAM} \rightarrow \mathsf{ASRAM} \rightarrow \mathsf{ASRT}^2_2$*,*
(2) SRAM *does not imply* ACA_0*, and* SRT^2_2 *does not imply* SRAM*.*

Proof. Clearly, $\mathsf{SRAM} \rightarrow \mathsf{SRT}^2_2$ and $\mathsf{ASRAM} \rightarrow \mathsf{ASRT}^2_2$. As for the implications $\mathsf{SRT}^2_2 \rightarrow \mathsf{ASRT}^2_2$ and $\mathsf{SRAM} \rightarrow \mathsf{ASRAM}$, these follow from the preceding lemma and the fact that SRT^2_2, and hence also SRAM, implies $\mathsf{B}\Pi^0_1$ ([3], comments after Definition 6.4, and Lemma 10.6). That $\mathsf{ACA}_0 \rightarrow \mathsf{SRAM}$ amounts to a formalization of the fact that $\mathbf{0}'$ is an s-Ramsey degree, and is straightforward.

We now prove (2). By relativizing Corollary 5.1.7 of Mileti [16], we get that for any set $X \not\geq_T \emptyset'$ there is set $Y \geq_T X$ such that $Y \not\geq_T \emptyset'$ and Y is s-Ramsey relative to X (i.e., computes an infinite homogeneous set for every X-computable stable coloring). Iterating, we thus obtain a sequence $Y_0 \leq_T Y_1 \leq_T \cdots$ such that $Y_e \not\geq_T \emptyset'$ and Y_{e+1} is s-Ramsey relative to Y_e for every e. Then the ideal $\{S : (\exists e)[S \leq_T Y_e]\}$ is clearly an ω-model of SRAM containing no set of degree $\mathbf{0}'$, and hence not a model of ACA_0. That SRT^2_2 does not imply SRAM is because the former has an ω-model consisting entirely of low_2 sets by relativizing and iterating Theorem 1.2, whereas the latter does not by Theorem 3.2 (2). □

The next result establishes a certain degree of similarity between ASRT^2_2 and SRT^2_2. In particular, we see that ASRT^2_2 is not overly weak by comparison with at least some of the principles studied in conjunction with SRT^2_2. The proof resembles that of Theorem 2.4 of [8] in that it uses the result that every effectively immune set computes a DNR function (see [11], p. 199)). Here we also need the fact, due to Kučera, that every 1-random set is effectively bi-immune ([15], Theorem 6).

Proposition 5.6. *Over* RCA_0*,* ASRT^2_2 *implies* DNR *but is not implied by* WKL_0*.*

Proof. For the implication, we give only an argument for ω-models, as it, and all the results it employs, admit straightforward formalization in RCA_0. So let $\mathscr{M}$ be an ω-model of ASRT^2_2 and fix $X \in \mathscr{M}$. Fix u as in the proof of Proposition 2.7, let $\widetilde{M} = \Phi^{X'}_u$, and let $\{\widetilde{M}_s\}_{s \in \omega}$ be an X-computable approximation of $\widetilde{M}$, sped

up to ensure that $2\widetilde{M}_s(\sigma) = \widetilde{M}_s(\sigma 0) + \widetilde{M}_s(\sigma 1)$ for all σ and s. If we define M by $M(\sigma, s) = \widetilde{M}_s(\sigma)$ for all σ and s, then $M \in \mathscr{M}$ and is a martingale approximation, so there exists a stable X-computable coloring $f \in \mathscr{M}$ and an infinite set $H \in \mathscr{M}$ such that M does not succeed on f and H is homogeneous for f. If we let $A = \{x : \lim_s f(x, s) = 1\}$ then $\widetilde{M}$ does not succeed on A, so A is X-random and hence effectively bi-immune relative to X. Then H, being an infinite subset or cosubset of A, is effectively immune relative to X, and so computes a DNR^X function $g \in \mathscr{M}$.

For the nonimplication, recall that for every incomplete Δ^0_2 PA degree $\mathbf{d}$ there exists an ω-model of WKL_0 consisting only of sets of degree below $\mathbf{d}$ (this is easily constructed using the fact that the PA degrees are dense; see Simpson [18], Theorem 6.5). Let $\mathscr{M}$ be any such model. By Theorem 1.5, $\mathbf{d}$ is not almost s-Ramsey, and so there is a Δ^0_2 martingale $\widetilde{M}$ which succeeds on every Δ^0_2 set containing an infinite subset or cosubset of degree at most $\mathbf{d}$. Let $\{\widetilde{M}_s\}_{s\in\omega}$ be a (suitably sped up) computable approximation to $\widetilde{M}$, and define a martingale approximation $M \in \mathscr{M}$ from it as above. Since all stable colorings in $\mathscr{M}$ that have an infinite homogeneous set in $\mathscr{M}$ have one of degree below $\mathbf{d}$, it follows that M succeeds on them all. Thus, $\mathscr{M}$ is not a model of ASRT^2_2. □

It follows that neither DNR nor COH imply ASRT^2_2 either, the latter because COH does not imply DNR by Theorem 3.7 of [8].

In view of the remarks made at the beginning of the section, it is natural to ask whether ASRT^2_2 implies WKL_0 or COH (the preceding proposition makes the first of these at least plausible). We conclude this section by giving negative answers to both questions.

Proposition 5.7. *Over* RCA_0*,* ASRT^2_2 *does not imply* WKL_0*.*

Proof. Let L be a given low 1-random set, and let $e \in \omega$ be given. If $\Phi_e^{\emptyset'}$ is a total martingale, let M, A, B and C be as in the proof of Proposition 2.7 with i a lowness index for L. Then $A = B \oplus C$, the set $L \oplus B$ is low, and $B \notin S[\Phi_e^{\emptyset'}]$. Furthermore, A is not in $S[M]$ and is therefore L-random, so, by van Lambalgen's theorem relative to L, B is L-random too. Since L is 1-random, another application of van Lambalgen's theorem yields that $L \oplus B$ is 1-random. By iterating, we can thus obtain an increasing sequence of sets $L_0 \leq_T L_1 \leq_T \cdots$ such that each L_e is low, 1-random, and computes a set $B \notin S[\Phi_e^{\emptyset'}]$ when $\Phi_e^{\emptyset'}$ is a total martingale.

We let $\mathscr{M}$ be the ideal $\{S : (\exists e)[S \leq_T L_e]\}$ and claim first of all that it is a model of ASRT^2_2. Indeed, suppose that $M \in \mathscr{M}$ is a martingale approximation. Then $\widetilde{M} : 2^{<\omega} \to \mathbb{Q}^{\geq 0}$ defined by $\widetilde{M}(\sigma) = \lim_s M(\sigma, s)$ for all σ is a $\Delta^{0,M}_2$ martingale and hence a Δ^0_2 martingale since every element in $\mathscr{M}$ is low. We can thus fix an e so that $\widetilde{M} = \Phi_e^{\emptyset'}$. Then by construction, L_e computes an infinite Δ^0_2 set $B \notin S[\widetilde{M}]$, say with computable approximation $\{B_s\}_{s\in\omega}$. If we define f by $f(x, s) = B_s(x)$ for all $x < s$, then f is a computable stable coloring, and hence $f \in \mathscr{M}$ and $f \leq_T M$. Clearly, M does not succeed on f in the sense of Definition 5.2, but B computes an infinite homogeneous set H for f, which, since $H \leq_T B \leq L_e$, belongs to $\mathscr{M}$.

Now recall that every ω-model of WKL_0 contains a set of PA degree, and that the class of these degrees is closed upwards (for the former, consider, e.g., the Π^0_1 class of all $\{0,1\}$-valued DNR functions, and see [5], Theorem 1.22.2; for the latter, see [5], Theorem 1.21.3). Also, every 1-random PA degree bounds $\mathbf{0}'$ by the main

result of Stephan [21]. So, as every element of $\mathscr{M}$ is Turing reducible to a low 1-random set, it follows that $\mathscr{M}$ cannot be a model of WKL_0. □

By Theorems 1.3 and 1.5 respectively, neither SRT^2_2 nor ASRAM has an ω-model consisting entirely of low sets. The same is true of COH because each of its ω-models must contain a p-cohesive set (see [3], p. 27), and each p-cohesive set has jump of degree strictly greater than $\mathbf{0}'$ by Theorem 2.1 of [12]. Hence, we immediately get the following:

Corollary 5.8. *Over* RCA_0*,* ASRT^2_2 *does not imply* SRT^2_2*,* ASRAM*, or* COH*.*

All the relations between the principles studied above are recapitulated in the following diagram (double arrows indicate implications whose reversals are not provable in RCA_0).

$$\begin{tikzcd}
 & \mathsf{ACA}_0 \arrow[d, Rightarrow] & \\
 & \mathsf{SRAM} \arrow[dl] \arrow[d, Rightarrow] & \\
\mathsf{ASRAM} \arrow[dr, Rightarrow] & \mathsf{SRT}^2_2 \arrow[d, Rightarrow] & \\
 & \mathsf{ASRT}^2_2 \arrow[r, leftrightarrow, "|" marking] \arrow[d, Rightarrow] & \mathsf{COH}, \mathsf{WKL}_0 \\
 & \mathsf{DNR} \arrow[d, Rightarrow] & \\
 & \mathsf{RCA}_0 &
\end{tikzcd}$$

We end by listing a few remaining questions concerning ASRAM and ASRT^2_2. Since SRT^2_2 has an ω-model consisting entirely of low$_2$ sets while SRAM does not, one of the first two would likely be answered by a solution to Question 4.4. The final question concerns the system WWKL_0, introduced in Simpson and Yu [24].

Question 5.9. Over RCA_0, does ASRAM imply SRAM? Does SRT^2_2 imply ASRAM or conversely? Does ASRT^2_2 imply WWKL_0?

WWKL_0 follows from WKL_0, and so cannot imply ASRT^2_2 by Proposition 5.6. Since the ω-models of WWKL_0 are precisely those that for every set X in them contain also an X-random ([1], Lemma 1.3 (2)), a negative solution to the last question may follow from showing that the collection of Δ^0_2 sets having an infinite subset or cosubset not computing any 1-randoms is not Δ^0_2 null. It is worth remarking that Kjos-Hanssen [14] (see also [13], Theorem 7.4) has recently proved the non-effective version of this, showing that almost every infinite subset of ω has an infinite subset not computing any 1-randoms.

References

[1] Klaus Ambos-Spies, Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A. Slaman. Comparing DNR and WWKL. *J. Symbolic Logic*, 69(4):1089–1104, 2004.

[2] Stephen Binns and Stephen G. Simpson. Embeddings into the Medvedev and Muchnik lattices of Π^0_1 classes. *Arch. Math. Logic*, 43(3):399–414, 2004.

[3] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *J. Symbolic Logic*, 66(1):1–55, 2001.

[4] Rod Downey, Denis R. Hirschfeldt, Joseph S. Miller, and André Nies. Relativizing Chaitin's halting probability. *J. Math. Log.*, 5(2):167–192, 2005.

[5] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic randomness and complexity*. Springer-Verlag, to appear.

[6] Rodney G. Downey, Denis R. Hirschfeldt, Steffen Lempp, and Reed Solomon. A Δ^0_2 set with no infinite low subset in either it or its complement. *J. Symbolic Logic*, 66(3):1371–1381, 2001.

[7] Damir D. Dzhafarov. *Combinatorics and computability theory*. PhD thesis, University of Chicago, in preparation.

[8] Denis R. Hirschfeldt, Carl G. Jockusch, Jr., Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A. Slaman. The strength of some combinatorial principles related to Ramsey's theorem for pairs. In *Computational prospects of infinity. Part II. Presented talks*, volume 15 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 143–161. World Sci. Publ., Hackensack, NJ, 2008.

[9] Denis R. Hirschfeldt and Sebastiaan A. Terwijn. Limit computability and constructive measure. In *Computational prospects of infinity. Part II. Presented talks*, volume 15 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 131–141. World Sci. Publ., Hackensack, NJ, 2008.

[10] Carl G. Jockusch, Jr. Ramsey's theorem and recursion theory. *J. Symbolic Logic*, 37:268–280, 1972.

[11] Carl G. Jockusch, Jr. Degrees of functions with no fixed points. In *Logic, methodology and philosophy of science, VIII (Moscow, 1987)*, volume 126 of *Stud. Logic Found. Math.*, pages 191–201. North-Holland, Amsterdam, 1989.

[12] Carl G. Jockusch, Jr. and Frank Stephan. A cohesive set which is not high. *Math. Logic Quart.*, 39(4):515–530, 1993.

[13] Bjørn Kjos-Hanssen. In *Computability, Reverse Mathematics, and Combinatorics: Open Problems*, Banff International Research Station (BIRS), pages 12–16. Alberta, Canada, 2009. `http://robson.birs.ca/~08w5019/problems.pdf/`.

[14] Bjørn Kjos-Hanssen. A law of weak subsets. To appear.

[15] Antonín Kučera. Measure, Π^0_1-classes and complete extensions of PA. In *Recursion theory week (Oberwolfach, 1984)*, volume 1141 of *Lecture Notes in Math.*, pages 245–259. Springer, Berlin, 1985.

[16] Joseph R. Mileti. *Partition Theorems and Computability Theory*. PhD thesis, University of Illinois at Urbana-Champaign, 2004.

[17] Claus-Peter Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitstheorie*. Lecture Notes in Mathematics, Vol. 218. Springer-Verlag, Berlin, 1971.

[18] Stephen G. Simpson. Degrees of unsolvability: a survey of results. In J. Barwise, editor, *Handbook of mathematical logic*, pages 631–652. North-Holland, Amsterdam, 1977.

[19] Stephen G. Simpson. *Subsystems of second order arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1999.

[20] Robert I. Soare. *Recursively enumerable sets and degrees*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987. A study of computable functions and computably generated sets.

[21] Frank Stephan. Martin-Löf random and PA-complete sets. In *Logic Colloquium '02*, volume 27 of *Lect. Notes Log.*, pages 342–348. Assoc. Symbol. Logic, La Jolla, CA, 2006.

[22] Sebastiaan A. Terwijn. *Computability and measure*. PhD thesis, Institute for Logic, Language, and Computation, 1998.

[23] Sebastiaan A. Terwijn. On the quantitative structure of Δ^0_2. In *Reuniting the antipodes—constructive and nonstandard views of the continuum (Venice, 1999)*, volume 306 of *Synthese Lib.*, pages 271–283. Kluwer Acad. Publ., Dordrecht, 2001.

[24] Xiaokang Yu and Stephen G. Simpson. Measure theory and weak König's lemma. *Arch. Math. Logic*, 30(3):171–180, 1990.

University of Chicago
E-mail address: damir@math.uchicago.edu